Math = Fun

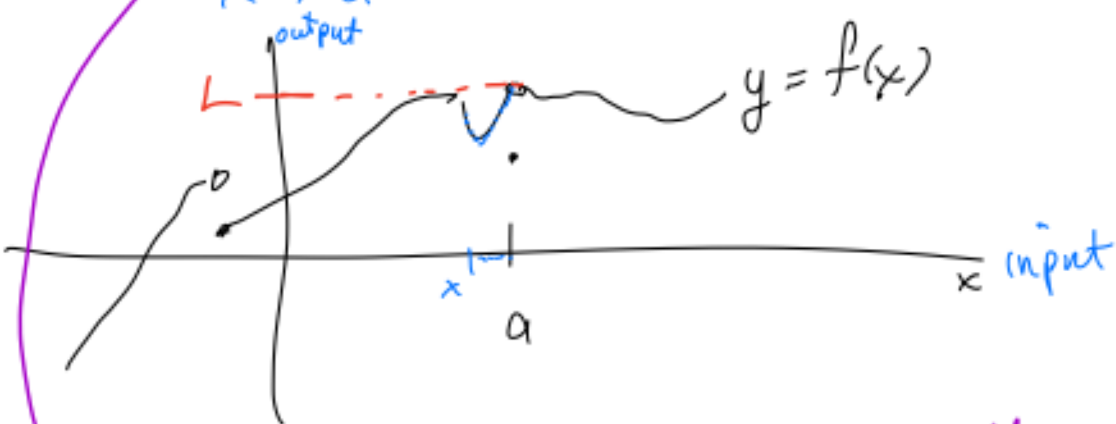
Calculus = study of FUNctions

Calc 2 = 2 • (Fun of Calc 1)

Important concepts of Calculus:

① Limit of a function at a point

$$\lim_{x \to a} f(x) = L$$



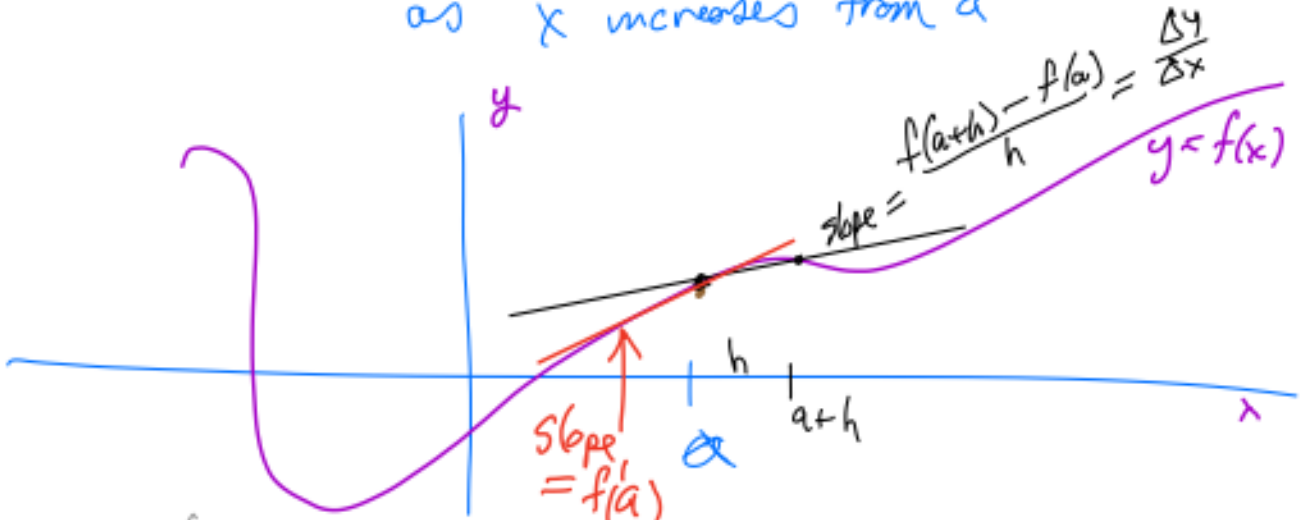

L = the number that f(x) is getting very close to as x gets very close to a (but not equal to a).

(1b) $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

② Derivative

$$f'(a) = \frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

= instantaneous rate of change of $f$ as $x$ increases from $a$




derived formulas for derivatives of all the standard function.

$(\sin(x))' = \cos(x)$, $(e^x)' = e^x$,

$(2^x)' = (\ln 2)\, 2^x$, $(\ln x)' = \frac{1}{x}$

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Example Find $g'(1)$ if

$$g(x) = x^2 \arctan(x)$$

$$\Rightarrow g'(x) = (x^2)' \arctan(x) + x^2 \cdot (\arctan(x))'$$

$$\Rightarrow g'(x) = 2x\arctan(x) + x^2\cdot\frac{1}{1+x^2}$$

$$g'(1) = 2\underbrace{\arctan(1)}_{\text{angle whose tangent is }(1)} + 1\cdot\frac{1}{1+1}$$

$$= \cancel{2}^{\,1}\frac{\pi}{\cancel{4}_{\,2}} + \frac{1}{2} = \boxed{\frac{\pi+1}{2}}$$

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Aside: Why is $\left(\arctan(x)\right)' = \dfrac{1}{1+x^2}$ ?

$$\tan\left(\arctan(x)\right) = x$$

(take deriv of both sides) $\tan'\left(\arctan(x)\right)\left[\arctan'(x)\right] = 1.$

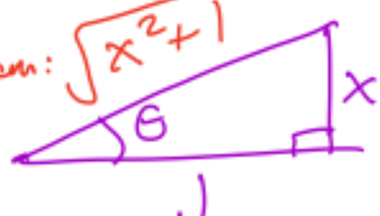
$$(\tan u)' = \sec^2(u)$$

$$\Rightarrow \sec^2\left(\arctan(x)\right)\cdot\arctan'(x) = 1$$

$$\arctan'(x) = \frac{1}{\sec^2\underbrace{\left(\arctan(x)\right)}_{\theta}} = \underline{\cos^2\left(\arctan(x)\right)}$$

$$= \cos^2(\theta)$$

Pyth theorem: $\sqrt{x^2+1}$




$\theta = \arctan(x)$ angle whose tangent is $x$

$$\tan\theta = x$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{x^2+1}} \Rightarrow \cos^2(\theta) = \frac{1}{x^2+1}$$

$$\Rightarrow \boxed{\arctan'(x) = \frac{1}{x^2+1}.}$$

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Derivative — useful for calculating rates of change, slope. velocity, fluid flow rate.

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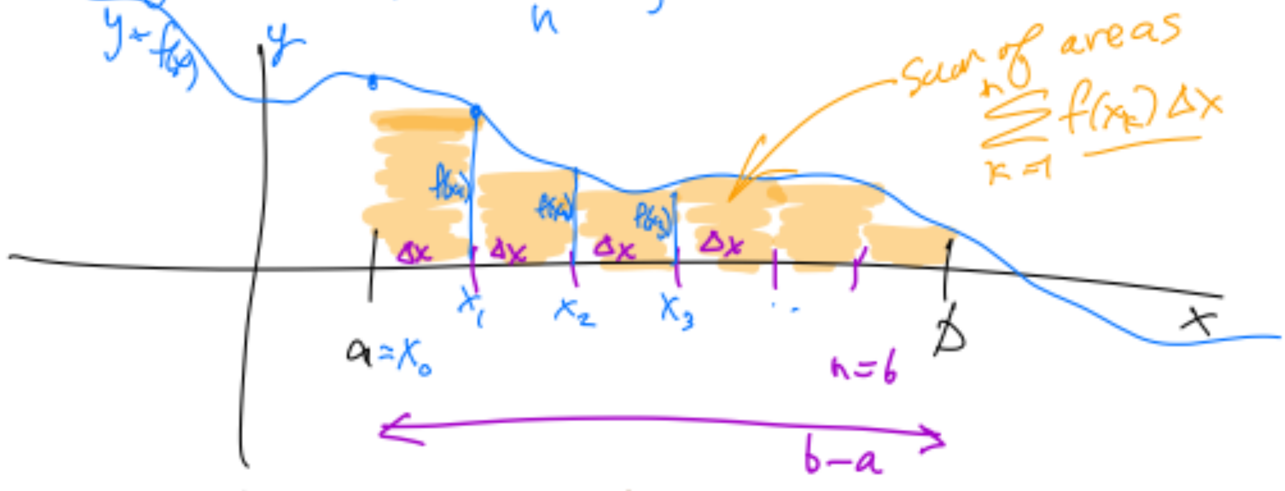
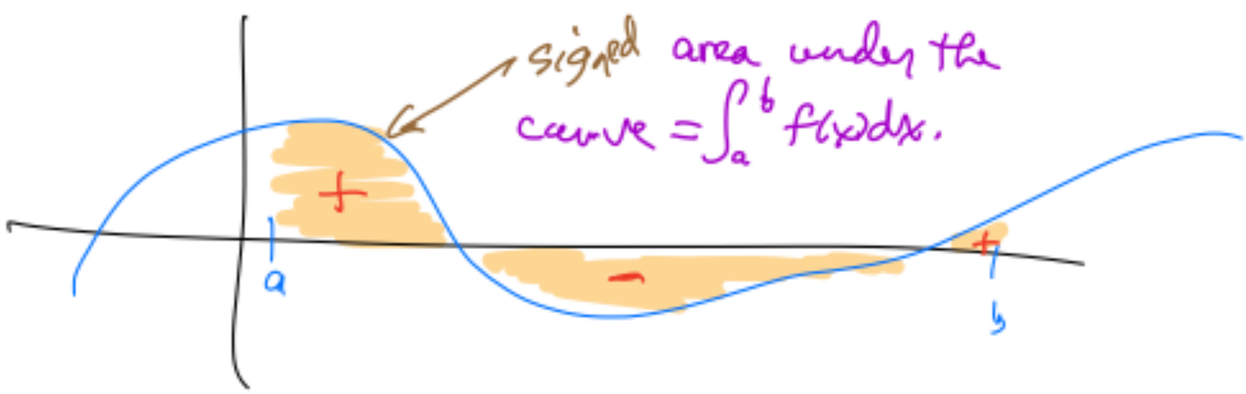
## ③ Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \overbrace{\sum_{k=1}^{n} f(x_k)\Delta x}^{f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x},$$

where $\Delta x = \dfrac{b-a}{n}$, and $x_k = a + k\Delta x$.




signed area under the curve $= \int_a^b f(x)dx.$



Connection between derivatives & integrals:

FUNdamental Theorem of Calculus:

$$\int_a^b \underbrace{F'(x)\,dx}_{dF} = F(x)\Big|_a^b = F(b) - F(a)$$

$$\underbrace{\int}_{\text{sum}} \underbrace{\begin{pmatrix}\text{rate of change}\\ \text{of } F(x)\end{pmatrix}\begin{pmatrix}\text{change}\\ \text{in}\\ x\end{pmatrix}}_{\text{small chge in } F(x).} = \begin{pmatrix}\text{total change in}\\ F(x)\end{pmatrix}$$

It's important for us to be able to calculate antiderivatives (finding F to make the F'), so that we can solve for the definite integrals.

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Practice Definite Integrals:

① $\int_0^1 \frac{1}{x+1}\,dx$ ② $\int_0^1 \frac{1}{x^2+1}\,dx$ ③ $\int_0^1 \frac{1}{2x^2+1}\,dx$

④ $\int_0^1 \frac{1}{2x^2+2x+1}\,dx$ ⑤ $\int_0^1 \frac{x}{2x^2+1}\,dx$ ⑥ $\int_0^1 \frac{1}{\sqrt{x}+1}\,dx$

① $\int_0^1 \frac{1}{x+1}\,dx = \ln|x+1| \Big|_0^1 = \ln(2) - \ln(1)$

$(\ln|x+1|)' = \frac{1}{x+1}$

$e^0 = 1$

$= \boxed{\ln(2)}$

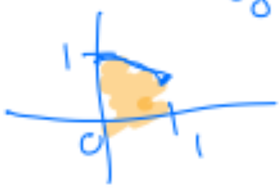
Note: $(\ln(x))' = \frac{1}{x}$, x positive

$(\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$, x negative.

combined $(\ln|x|)' = \frac{1}{x}$.

② $\int_0^1 \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_0^1 = \arctan(1) - \arctan(0)$

$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$



③ $\int_{x=0}^{x=1} \frac{1}{2x^2+1}\,dx = \int_{\substack{x=0\\u=1}}^{u=3} \frac{1}{u}\frac{du}{4x}$ We tried!

not able to get rid of x.

$u = 2x^2+1 \Rightarrow du = 4x\,dx$

$u^2+1 = 2x^2+1$

$u = \sqrt{2}\,x \Rightarrow du = \sqrt{2}\,dx$

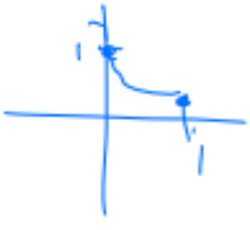
$u^2+1 = (\sqrt{2}x)^2+1 = 2x^2+1$

$dx = \frac{1}{\sqrt{2}}du$

$= \int_{\substack{x=0\\u=0}}^{\substack{u=\sqrt{2}\\x=1}} \frac{1}{u^2+1}\frac{du}{\sqrt{2}} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{1}{u^2+1}\,du$

$= \frac{1}{\sqrt{2}}\arctan(u)\Big|_0^{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}) - \underbrace{\arctan(0)}_{0}\right)$

$= \boxed{\frac{1}{\sqrt{2}}\arctan(\sqrt{2})}.$



Aside: Differential forms $3\cos(x)\,dx$, $du$

$\to$ what do they tell us?

Example $y = x^2$


examine $y = x^2$ near $x = 1$

differential

$dy = d(x^2) = 2x\,dx$

Go from $x=1$ to $x = 1.1$

$y(1)$ vs. $y(1.1)$:

$dy = \text{change in } y \approx 2x\,dx = 2 \cdot 1 \cdot (.1)$

$= .2$

According to $dy = 2x\,dx$, $y(1) = 1$

(change in $y$) $\approx .2$ ie $y(1.1) \approx 1.2$

Check $y(1.1) = 1.1^2 = 1.21$

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(5) $\displaystyle\int_0^1 \frac{x}{2x^2+1}\,dx = \int_{\substack{u=1\\ x=0}}^{u=3} \frac{\cancel{x}}{u}\left(\frac{du}{4\cancel{x}}\right)$

let $u = 2x^2+1$

$du = 4x\,dx$

$dx = \dfrac{du}{4x}$

$\displaystyle = \frac{1}{4}\int_{u=1}^{3} \frac{du}{u} = \frac{1}{4} \ln|u| \Big|_1^3$

$= \frac{1}{4}\left(\ln|3| - \ln|1|\right) = \boxed{\dfrac{\ln(3)}{4}}$

⑥ $\int_0^1 \frac{1}{\sqrt{x}+1}\,dx = \int_{x=0}^{x=1} \frac{1}{u}(2\sqrt{x})\,du = \int_{\substack{x=0\\u=1}}^{u=2,\,x=1} \frac{1}{u}\,2(u-1)\,du$

$u = \sqrt{x}+1 = x^{1/2}+1 \Rightarrow \sqrt{x} = u-1$

$du = \frac{1}{2\sqrt{x}}\,dx \qquad dx = (2\sqrt{x})\,du$

$= 2\int_1^2 \frac{u-1}{u}\,du = 2\int_1^2 \left(1-\frac{1}{u}\right)du$

$\frac{u}{u}-\frac{1}{u}$

$= 2\cdot\left(u-\ln|u|\right)\Big|_1^2$

$= 2\left[\left(2-\ln|2|\right)-\left(1-\underbrace{\ln|1|}_{0}\right)\right]$

$= 2\left(1-\ln(2)\right)$

$= \boxed{2-2\ln(2)}.$